THE OUTER AUTOMORPHISM GROUP OF AN OLIGOMORPHIC GROUP IS T.D.L.C.

ANDRÉ NIES

Abstract. A closed subgroup of Sym($\mathbb{N}$) is called oligomorphic if for each k, the canonical action on $\mathbb{N}^{[k]}$ has only finitely many orbits. The group Aut($\mathbb{Q}$, <) is an example. Oligomorphic groups are in a sense opposite to t.d.l.c. groups. I will report on an unexpected connection.

Each oligomorphic group G is Roelcke precompact, namely, each open subgroup has only finitely double cosets. For such a G, $\mathrm{Aut}(G)$ carries a natural Polish topology. We show that $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$. Thus $\mathrm{Out}(G)$ is also a Polish (in fact, non-Archimedean) group. Next we show that for oligomorphic G, $\mathrm{Out}(G)$ is t.d.l.c. Joint work with G. Paolini.

These are notes for the talk André Nies gave at a 2024 conference in honour of George Willis' birthday.

By $\mathrm{Aut}(G)$ we will denote the group of (bi-)*continuous* automorphisms of a Polish group G. The result that $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is t.d.l.c. for each oligomorphic group G is [8, Theorem 3.3]. We use arguments centred on non-Archimedean topological groups, with only a slight amount of model theory.

I will first give background and explain the terms in the result, and then proceed to the proof.

1. Is there a Polish topology on $\mathrm{Aut}(G)$?

A Polish space is a separable topological space with a topology that is induced by a complete metric. A Polish group G is a topological group based on a Polish space.

Question 1.1. *Given a Polish group G, can one topologise* $\mathrm{Aut}(G)$ *as a Polish group so that the action* $\mathrm{Aut}(G) \times G \to G$ *is continuous?*

We give a criterion to answer this in the affirmative for several classes of Polish groups. Suppose we can assign to G a countable basis $\mathcal{B}(G)$ for the topology such that $\mathcal{B}(G)$ is invariant under the action of $\mathrm{Aut}(G)$. Then the action yields an injective homomorphism $\mathrm{Aut}(G) \to \mathrm{Sym}(\mathcal{B}(G))$, where $\mathrm{Sym}(X)$ is the group of permutations on X. The group $\mathrm{Sym}(X)$ has the topology of pointwise convergence: the pointwise stabilisers of finite subsets of X form a neighbourhood basis of the identity, consisting of open subgroups.

Fact 1.2. Suppose the range of the embedding $\mathrm{Aut}(G) \to \mathrm{Sym}(\mathcal{B}(G))$ is closed. Then the answer to Question 1.1 is in the affirmative, via the topology on $\mathrm{Aut}(G)$ that makes the embedding a homeomorphism. Thus, one declares as open the subgroups of the form

$$\{\Phi \in \mathrm{Aut}(G) \colon \forall i = 1 \ldots n \ \Phi(A_i) = A_i\},$$

where $A_1, \ldots, A_n \in \mathcal{B}(G)$.

Proof. Note that the induced topology on $\mathrm{Aut}(G)$ is Polish (in fact, it is non-Archimedean). An action of a Polish group on a Polish space is continuous iff it is separately continuous (see [5]). So it suffices to show that for each $g \in G$, the map $\mathrm{Aut}(G) \to G$ given by $\Phi \mapsto \Phi(g)$ is continuous. Suppose then that $\Phi(g) \in B$ where $B \in \mathcal{B}(G)$. Let $A = \Phi^{-1}(B)$. Then $g \in A$. If $\Theta \in \mathrm{Aut}(G)$ is such that $\Theta(A) = \Phi(A)$, then $\Theta(g) \in B$. □

From now on we restrict to the context of infinite closed subgroups of $\mathrm{Sym}(\mathbb{N})$; by G, H we usually denote such groups. Fig. 1 displays some Borel classes that are invariant under conjugation in $\mathrm{Sym}(\mathbb{N})$, where the arrows denote inclusion between classes. Prop. 1.6 yields an affirmative

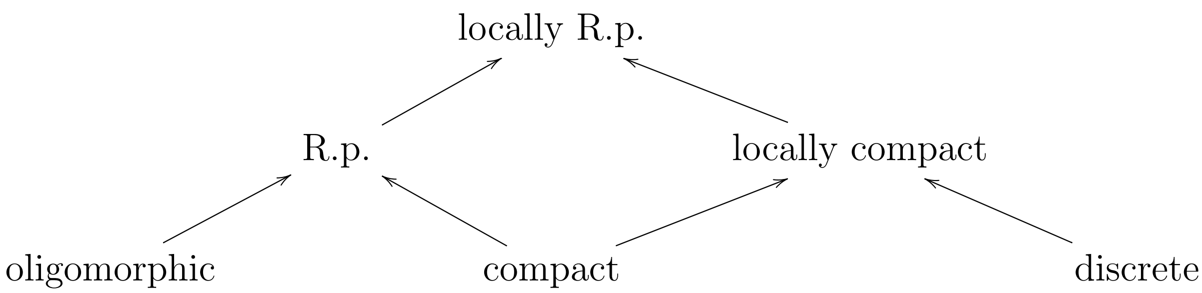

FIGURE 1. Some Borel classes of infinite closed subgroups of $\mathrm{Sym}(\mathbb{N})$

answer to Question 1.1 for groups in the largest class (and thus all subclasses). For each G in this class, we find an $\mathrm{Aut}(G)$-invariant basis $\mathcal{B}(G)$ such that the range of the embedding $\mathrm{Aut}(G) \to \mathrm{Sym}(\mathcal{B}(G))$ is closed.

Definition 1.3. (i) A group G is *Roelcke precompact (R.p.)* if each open subgroup U has only finitely many double cosets UgU, where $g \in G$.
(ii) G is *locally Roelcke precompact* if it has a R.p. open subgroup.

Trivially, each open subgroup of a R.p. group is also R.p. Thus, if G is locally R.p., its R.p. open subgroups form a neighbourhood basis of the identity. For background on (locally) Roelcke precompact groups, also outside the class of non-Archimedean groups, see Rosendal [9] as well as Zielinski [12].

Definition 1.4 (Cameron [1])**.** A closed subgroup G of $\mathrm{Sym}(\mathbb{N})$ is *oligomorphic* if for each n, the action of G on $\mathbb{N}^n$ has only finitely many orbits.

Fact 1.5 ([10], Theorem 2.4)**.** *A closed subgroup G of* $\mathrm{Sym}(\mathbb{N})$ *is oligomorphic iff G has only finitely many* 1*-orbits and G is Roelcke precompact.*

Note that being oligomorphic is a permutation group notion, while the other notions are topological. A group G (always closed subgroup of $\mathrm{Sym}(\mathbb{N})$) is called quasi-oligomorphic [6] if it is homeomorphic to an oligomorphic group. The topological inverse limits of such groups are up to homeomorphism the Roelcke precompact groups [10].

The following establishes a Polish topology on $\mathrm{Aut}(G)$ for locally R.p. G.

Proposition 1.6. *Given a locally R.p. group G, let $\mathcal{B}(G)$ be the set of cosets of Roelcke precompact open subgroups, together with $\emptyset$. This is a countable basis for G. The range of the embedding in Fact 1.2 is closed.*

Proof. To show $\mathcal{B}(G)$ is countable, it's enough to verify that it contains only countably many subgroups. Each such subgroup U is the union of finitely many double cosets of the point stabiliser of a finite subset of $\mathbb{N}$, so there are only countably many possibilities for U.

To show the range of the embedding is closed, we need a structure $\mathcal{W}(G)$ (in the sense of model theory) with domain $\mathcal{B}(G)$ so that the range equals $\mathrm{Aut}(\mathcal{W}(G))$. Take the intersection operation, together with the partial binary operation $A \cdot B$ that is defined when A is a left coset and B a right coset of the same subgroup U, and then it is the usual product in G. For the verification that this works see [3, Section 3]. The structure $\mathcal{W}(G)$ is called the *meet groupoid* of G. □

Remark 1.7. For [quasi]-oligomorphic groups, every open subgroup is R.p. For t.d.l.c. groups G, the R.p. open subgroups coincide with the compact open ones. So the domain of the meet groupoid $\mathcal{W}(G)$ consists of the compact open cosets.

For a locally compact group G, the group $\mathrm{Aut}(G)$ carries the *Braconnier topology.* It is given by neighbourhoods of the identity of the form $C(K, U)$, for any compact K and open nbhd U of 1; here

$$\alpha \in C(K, U) \text{ iff } \alpha(x) \in Ux \wedge \alpha^{-1}(x) \in Ux.$$

If G is discrete, then the Braconnier topology coincides with the topology of pointwise convergence on $\mathrm{Aut}(G)$, because we can assume that U is the trivial group, and of course each compact set is finite. The Braconnier topology is the coarsest topology making the action $\mathrm{Aut}(G) \times G \to G$ continuous [2, Appendix I]. Since two Polish topologies on a group coincide in case one is contained in the other [5, 2.3.4], this implies that the Braconnier topology coincides with the one given by Prop 1.6. A direct proof of this fact is in [3, Section 9]. (It would be interesting to follow up on this connection for the remaining classes of groups in Figure 1.)

2. When is $\operatorname{Inn}(G)$ closed in $\operatorname{Aut}(G)$?

The group of inner automorphisms $\operatorname{Inn}(G)$ is a normal subgroup of $\operatorname{Aut}(G)$. It is of interest whether it is closed in $\operatorname{Aut}(G)$, for in that case the outer automorphism group $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ is a Polish group.

Wu [11] gave an example of a discrete group L such that $\operatorname{Inn}(L)$ is not closed in $\operatorname{Aut}(L)$. A slight modification yields a nilpotent-2 group of exponent 3. (Curiously, this group is finite automaton presentable, and in fact similar to the groups recently studied in [7].)

Example 2.1 (Similar to Example 4.5 in [11])**.** Let G be generated by elements of a_i, b_i, c order 3 ($i \in \mathbb{N}$), where c is central, with the relations $b_i a_i b_i^{-1} = a_i c$, $[a_i, a_k] = [a_i, b_k] = [b_i, b_k] = 1$ for $i \neq k$. An automorphisms Φ of L is given by $\Phi(a_i) = a_i c$, $\Phi(b_i) = b_i$, and $\Phi(c) = c$. It is not inner, but in the closure of $\operatorname{Inn}(G)$.

Proof. To check that Φ is indeed in $\operatorname{Aut}(G)$, note that c can be omitted from the list of generators. Given a word w in a_i, b_k where each letter occurs with exponent 1 or 2, use that $\Phi(w) = wc^{k \bmod 3}$ where k is the number of occurrences of a_i's. The inverse of Φ is given by $a_i \mapsto a_i c^{-1}$ and the rest as before.

Write $g_n = \prod_{i<n} b_i$. We have $g_n a_i g_n^{-1} = a_i c$ for each $i < n$, and $g_n b_i g_n^{-1} = b_i$ for each i. Letting Φ_n be conjugation by g_n, we have $\lim_n \Phi_n = \Phi$ in $\operatorname{Aut}(G)$. For each $g \in G$, conjugation by g fixes almost all the generators. So Φ is not inner. ☐

So that's *dis*couraging, but things look brighter on the left side of Figure 1.

Theorem 2.2 (extends Thm. 3.3 in [8])**.** $\operatorname{Inn}(G)$ *is closed in* $\operatorname{Aut}(G)$*, for each Roelcke precompact group closed subgroup G of* $\operatorname{Sym}(\mathbb{N})$*.*

Proof. We follow the proof for oligomorphic groups [8, Theorem 3.3(i)] The open cosets of G form a groupoid (a category where all morphisms are invertible). For subgroups U, V of G, by $\operatorname{Mor}(U, V)$ we denote the set of right cosets of U that are left cosets of V.

Claim 1. $\operatorname{Mor}(U, V)$ *is finite, for any open subgroups U, V of G.*

To see this, first note that $\operatorname{Mor}(U, U)$ is finite because each coset in it is a double coset of U. Now suppose that there is a $B \in \operatorname{Mor}(U, V)$. There is a bijection $\operatorname{Mor}(U, U) \to \operatorname{Mor}(U, V)$ via $A \mapsto A \cdot B$. Thus $\operatorname{Mor}(U, V)$ is finite as required. This shows the claim.

A fact in the theory of Polish groups states that each G_δ subgroup of a Polish group is closed (see [5, Prop. 2.2.1]); this relies on the Baire category theorem. So it suffices to show that $\mathrm{Inn}(G)$ is a G_δ subset of $\mathrm{Aut}(G)$. Let $(A_n)_{n\in\omega}$ be a listing of the open cosets of G without repetition. Given $\Phi \in \mathrm{Aut}(G)$, we will define a set T_Φ of strings over some infinite alphabet which is closed under prefixes, and thus can be seen as a rooted tree. The alphabet consists of pairs of open cosets of G. To define the n-th level we think o approximating some $g \in G$ such that $gA_ig^{-1} = \Phi(A_i)$ for $i < n$. At the n-level we have certain pairs B_i, C_i of approximations, where $i < n$:

$$T_\Phi = \{\langle (B_i, C_i)\rangle_{i<n} \colon \Phi(A_i) = B_i \cdot A_i \cdot C_i^{-1} \wedge \bigcap_{i<n} (B_i \cap C_i) \neq \emptyset\}.$$

The domains of such open cosets B_i and C_i are determined by the condition that $\Phi(A_i) = B_i \cdot A_i \cdot C_i^{-1}$. By Claim 1, there are only finitely many possibilities for cosets with given left and right domain. So the tree T_Φ is finitely branching.

Claim 2. $\Phi \in \mathrm{Aut}(G)$ *is inner iff* T_Φ *has an infinite path.*

For the verification see Claim 2 in the proof of [8, Theorem 3.3(i)], which goes through for the general setting of R.p. groups. Assuming the claim, we conclude the argument as follows. By König's Lemma, T_Φ has an infinite path iff each of its levels (i.e., strings of a length n) is nonempty. Whether the n-th level of T_Φ is nonempty only depends on the values $\Phi(A_0), \dots, \Phi(A_{n-1})$, so the set of such Φ is open in $\mathrm{Aut}(G)$. Thus the condition that each level is nonempty is G_δ. So the claim shows that the subgroup $\mathrm{Inn}(G)$ is G_δ in $\mathrm{Aut}(G)$. □

Example 2.3. $G = \mathrm{Aut}(\mathbb{Q}, <)$ is oligomorphic. The map $\pi(x) = -x$ is in the normaliser N_G and not in G. A continuous automorphism of G not in $\mathrm{Inn}(G)$ is therefore given by $g \mapsto \pi^{-1} \circ g \circ \pi$. This is the only one up to $\mathrm{Inn}(G)$: it is known that $\mathrm{Out}(G) \cong N_G/G$ which has two elements [**?**, Cor 1.6].

Remark 2.4. If G is R.p., then the topology on $\mathrm{Inn}(G)$ inherited from $\mathrm{Aut}(G)$ is the expected one, namely the one given by quotient topology of G by the centre. For, the centre $Z = Z(G)$ is closed. We have a Polish topology on G/Z by declaring UZ/Z open iff UZ is open in G. The canonical isomorphism $L \colon G/Z \to \mathrm{Inn}(G)$ is continuous. Since $\mathrm{Inn}(G)$ is Polish as a closed subgroup of $\mathrm{Aut}(G)$, L is a homeomorphism by a standard result in the theory of Polish groups (see e.g., [5, 2.3.4]).

The following is [8, Theorem 2.1].

Theorem 2.5. *Let G be oligomorphic. Then N_G/G is profinite, where N_G is the normaliser of G in* $\mathrm{Sym}(\omega)$*.*

The result is proved using a model-theoretic technique: an oligomorphic G equals $\mathrm{Aut}(M)$ for the so-called *canonical structure* for G (made out of the n-orbits for all $n \geq 1$). This M is $\aleph_0$-categorical, which implies that the G-invariant relations coincide with the first-order definable relations. Also, $N_G = \mathrm{Aut}(\mathcal{E}_M)$ where $\mathcal{E}_M$ is a reduct of M, its so-called *orbital structure*. The proof which we omit here consists of showing that $\mathrm{Aut}(\mathcal{E}_M)/\mathrm{Aut}(M)$ is a profinite group.

By [4] together with [8], every separable profinite group can be realised as $\mathrm{Out}(G) = N_G/G$: the structure constructed in [4] has "no algebraicity", which implies by [8, Th. 4.7] that each automorphism of G is given as conjugation by a permutation $\pi \in N_G$.

Now let $R\colon N_G \to \mathrm{Aut}(G)$ be the homomorphism defined by $R(\pi)(g) = \pi \circ g \circ \pi^{-1}$. The following is part of [8, Theorem 3.3].

Theorem 2.6. *(i) The subgroup* $\mathrm{range}(R)$ *is open in* $\mathrm{Aut}(G)$.
(ii) The group $\mathrm{Out}(G)$ *is t.d.l.c., having the compact group* $\mathrm{range}(R)/\,\mathrm{Inn}(G)$ *as an open subgroup.*

Proof. **(i)**. For $b, a \in \omega$, we will write $[b, a]$ for the coset $\{g \in G : b = g(a)\}$. Note that for $a, b, c \in \omega$, if $[c, a] \supseteq [b, a] \neq \emptyset$ then $b = c$.

Let $a_1, \ldots, a_n$ represent the 1-orbits of G. It suffices to show that the open subgroup $\{\Phi \in \mathrm{Aut}(G) : \Phi(G_{a_i}) = G_{a_i} \text{ for each } i\}$ is contained in $\mathrm{range}(R)$. Suppose that Φ is in this subgroup. Since Φ fixes each subgroup G_{a_i}, for each coset $D = [b, a_i]$, $\Phi(D)$ is also a left coset of G_{a_i}. Hence, by **??**, $\Phi(D)$ can be written in the form $[d, a_i]$. Define a function $\pi_\Phi\colon \omega \to \omega$ by

$$\pi_\Phi(b) = d \text{ if } \Phi([b, a_i]) = [d, a_i], \text{ where } [b, a_i] \neq \emptyset.$$

Clearly π_Φ and $\pi_{\Phi^{-1}}$ are inverses. So $\pi_\Phi \in \mathrm{Sym}(\omega)$. The following establishes (i).

Claim. $\pi_\Phi \in N_G$ *and* $R(\pi_\Phi) = \Phi$.
Write $\pi = \pi_\Phi$. To verify the claim, we first show that $\Phi([r, s]) = [\pi(r), \pi(s)]$ for each $r, s \in \omega$ such that $[r, s] \neq \emptyset$. Note that by hypothesis on Φ and since $G_{a_i} = [a_i, a_i]$, we have $\pi(a_i) = a_i$ for each i. Also note that $\Phi([b, c])^{-1} = \Phi([c, b])$ for each $b, c \in \omega$. Let now i be such that r and s are in the 1-orbit of a_i. We have $[r, s] = [r, a_i] \cdot [a_i, s] = [r, a_i] \cdot [s, a_i]^{-1}$. So

$$\Phi([r, s]) = \Phi([r, a_i]) \cdot \Phi([s, a_i])^{-1} = [\pi(r), a_i] \cdot [\pi(s, a_i]^{-1} = [\pi(r), \pi(s)].$$

Next, for each $h \in G$ we have $\{h\} = \bigcap_{s \in \omega} hG_s$, and $hG_s = [h(s), s]$. Then $\{\Phi(g)\} = \bigcap_{s \in \omega} \Phi([g(s), s]) = \bigcap_{s \in \omega}[\pi(g(s)), \pi(s)] = \bigcap_{t \in \omega}[\pi(g(\pi^{-1}(t))), t] = \{g^\pi\}$, as required. This shows the claim.

(ii). The kernel of R is the centraliser C_G of G in $\mathrm{Sym}(\omega)$, which is a normal subgroup of N_G. This consists of the permutations that are definable in the canonical structure for G, so C_G is a finite. So GC_G is a normal subgroup of N_G. GC_G is also closed in $\mathrm{Sym}(\omega)$ because G has finite index in it. Since C_G is the kernel of R and $R(G) = \mathrm{Inn}(G)$, we have $R^{-1}(\mathrm{Inn}(G)) = GC_G$. Thus $\mathrm{range}(R)/\mathrm{Inn}(G)$ is topologically isomorphic to N_G/GC_G. By 2.6 N_G/G is compact. So N_G/GC_G is also compact as its topological quotient. □

To summarise, we have

$$\mathrm{Inn}(G) \cong G/Z(G) \leq_c N_G/C_G \leq_o Aut(G).$$

Taking the quotient by $\mathrm{Inn}(G)$, we get a t.d.l.c. group with its compact open subgroup predicted by van Dantzig's theorem:

$$N_G/GC_G \leq_o \mathrm{Out}(G).$$

3. Conclusion

Related work in progress with Philipp Schlicht started during his Auckland visit March-June 2024; it attempts to obtain the result in a purely model-theoretic way. The work in progress is based on the notion of bi-interpretations, and using them will hopefully lead to a better understanding of the structure of $\mathrm{Out}(G)$.

At present we know that $\mathrm{Out}(G)$ for oligomorphic G can be any separable profinite group [4]. We don't have an example where $\mathrm{Out}(G)$ is properly t.d.l.c., say discrete and infinite.

References

[1] P. Cameron. *Oligomorphic permutation groups*, volume 152. Cambridge University Press, 1990. 1.4

[2] P.-E. Caprace and N. Monod. Decomposing locally compact groups into simple pieces. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 150, pages 97–128. Cambridge University Press, 2011. 1

[3] A. Nies (editor). Logic Blog 2022. Available at `https://arxiv.org/pdf/2302.11853.pdf`, 2022. 1, 1

[4] D. Evans and P. Hewitt. Counterexamples to a conjecture on relative categoricity. *Annals of Pure and Applied Logic*, 46(2):201–209, 1990. 2, 3

[5] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009. 1, 1, 2, 2.4

[6] A. Nies, P. Schlicht, and K. Tent. Coarse groups, and the isomorphism problem for oligomorphic groups. *Journal of Mathematical Logic*, page 2150029, 2021. 1

[7] A. Nies and F. Stephan. Word automatic groups of nilpotency class 2. *Information Processing Letters*, 183:106426, 2024. 2

[8] G. Paolini and A. Nies. Two classes of oligomorphic groups with smooth topological isomorphism relation. *arXiv preprint arXiv:2410.02248*, 2024. (document), 2.2, 2, 2, 2

[9] C. Rosendal. *Coarse Geometry of Topological Groups*. Cambridge Tracts in Mathematics. Cambridge University Press, 2021. 1

[10] T. Tsankov. Unitary representations of oligomorphic groups. *Geometric and Functional Analysis*, 22(2):528–555, 2012. 1.5, 1

[11] Ta-Sun Wu. On (ca) topological groups, ii. *Duke Math. Journal*, 38:533–539, 1971. 2, 2.1

[12] J. Zielinski. Locally Roelcke precompact Polish groups. *Groups, Geometry, and Dynamics*, 15(4):1175–1196, 2021. 1